UNDERGRADUATE FOURTH SEMESTER (HONOURS) EXAMINATIONS, 2022

Subject: Mathematics **Course ID: 42113**

Course Code: SH/MTH/403/C-10

Course Title: Ring Theory and Linear Algebra - I

Time: 2 hours **Full Marks: 40**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. Answer *any five* of the following questions: **(2X5=10)**

a) Give examples of commutative and non-commutative rings containing infinitely many divisors of zero.

b) Suppose R is a ring with unity 1 and I is an ideal of R such that $1 \in I$. Prove that $I = R$.

c) Is $\{\bar{0}\}$ a prime ideal in $(\mathbb{Z}_6, +_6, \cdot_6)$? Justify your answer.

d) Find the kernel of the homomorphism $\varphi: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, given by $\varphi(a, b) = a, \ \forall\, a, b \in \mathbb{Z}$.

e) Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ over the field $(\mathbb{R}, +, \cdot)$. Examine whether $W = \{f \in V : f \text{ is odd function}\}$ is a subspace or not?

f) Find a basis of the subspace S of $\mathcal{M}_{2\times 2}(\mathbb{R})$, where S is the collection of all symmetric matrices.

g) Let T be a linear transformation between two finite dimensional vector spaces. If $\ker T$ is singleton, then show that T is injective.

h) Let V be the vector space of all real valued continuous functions on $[0,1]$ and $T: V \to \mathbb{R}$ be a function defined by $T(f) = \int_0^1 f(x)dx, \ \forall f \in V$. Check whether T is linear or not.

2. Answer *any four of* the following questions: **(5X4=20)**

a) Let $J = \{a + ib \in \mathbb{Z}[i] : a, b \in 7\mathbb{Z}\}$. Show that J is an ideal of $\mathbb{Z}[i]$. Is J a maximal ideal? Also find the number of elements of $\mathbb{Z}[i]/J$. **2+1+2**

b) (i) Prove that every Boolean ring is commutative.
(ii) Prove that characteristic of an integral domain is either zero or prime. **2+3**

c) State and prove the first isomorphism theorem for rings.

d) (i) Define change of coordinate matrix on a vector space. **2+3**

(ii) Let $\mathcal{P}_2(\mathbb{R})$ be the space of all polynomimals of degree upto 2 over the field $\mathbb{R}$. Find the coordinates of $a + bx + cx^2 \in \mathcal{P}_2(\mathbb{R})$ with respect to the bases $\{x^2, x, 1\}$and $\{1, 1 + x, 1 + x + x^2\}$.

e) Let V be a vector space over the field F and W be a subspace of V. Prove that

$\dim (V/W) = dim\, V - dim\, W$.

f) Let V and W be two vector spaces over the field $\mathbb{F}$ and $T: V \to W$ be a linear map. Show that T is one-one if and only if T transforms a linearly independent subset of V to a linealy independent subset of W.

3. Answer *any one* of the following questions: (10X1=10)

a) (i) Give an example of a prime ideal which is not maximal. Also give an example ofa maximal ideal which is not prime.

(ii)If $a, b$are two elements of a field $F$and $b \neq 0$, then prove that $a = 1$ if

$$(ab)^2 = ab^2 + bab - b^2.$$

(iii) Prove that the sum of two subspaces of a vector space is again a subspace.

(iv) Let Vbe the vector space of all 5×5 matrices over $\mathbb{C}$ and $B \in V$.Verify whether

$T: V \to V$,defined by $T(A) = AB - BA$ for all $A \in V$, is a linear map.

3+2+3+2

b) (i) Examine whether the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is an integral domain or not.

(ii) Find all ring homomorphisms from the ring $(\mathbb{Z}_{12}, +, .)$ to the ring $(\mathbb{Z}_{30}, +, .)$.

(iii) Construct two distinct linear maps from $\mathbb{R}^3$ to $\mathbb{R}^3$each of which has its range the subspace spanned by $(1,0,-1)$ and $(1,2,2)$.

(iv) Show that the reflection about the line $y = x$ on $\mathbb{R}^2$ is a linear map. **2+3+3+2**

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